

# Crowd-anticrowd model of the Minority Game

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## Abstract

We provide a theoretical description of the Minority Game in terms of crowd effects. The size of the fluctuations arising in the game is controlled by the interplay between crowds of like-minded agents and their anti-correlated partners (anticrowds). The theoretical results are in good agreement with the numerical simulations over the entire parameter range of interest.

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The study of agent-based models of complex adaptive systems is attracting much attention [1]. Among the many possible interdisciplinary applications is the growing field of econophysics [2]: each agent knows the past ups and downs in the index of a financial market and must decide how to trade based on this global information. The Minority Game (MG) introduced by Challet and Zhang [3,4], offers arguably the simplest paradigm for such a complex, adaptive system. The MG comprises an odd number  $N$  of agents, each equipped with  $s$  strategies and a memory size  $m$ , who repeatedly compete to be in the minority [3–10]. The most striking feature arising from numerical simulations is the non-monotonic variation in the size of the fluctuations (i.e. standard deviation) produced by the MG as  $m$  is varied [4]. Challet *et al* provided a sophisticated formal connection between the MG and spin glass systems [5] which offers many fascinating quantitative insights into the MG’s dynamics. Given the complexity of this dynamics, it is understandable that no general theory has yet been proposed which yields quantitative agreement with the numerical results of Ref. [4] over the full range of  $m$ ,  $s$  and  $N$  values.

In this paper, we show that a theoretical model can be constructed in a surprisingly simple way by incorporating the ‘crowd’ effects (i.e. strong inter-agent correlations) which arise within the interacting, many-agent population. The results yield good agreement with the numerical results [4] over the entire range of  $m$ ,  $s$  and  $N$ . The non-monotonic behaviour of the standard deviation [3,4] is shown to arise from a fascinating interplay between a crowd and its anti-correlated partner (‘anticrowd’).

The MG [3] comprises an odd number of agents  $N$  who choose repeatedly between option 0 (e.g. buy) and option 1 (e.g. sell). The winners are those in the minority group, e.g. sellers win if there is an excess of buyers. The outcome at each timestep represents the winning decision, 0 or 1. A common bit-string of the  $m$  most recent outcomes is made available to the agents at each timestep. The agents randomly pick  $s$  strategies at the beginning of the game, with repetitions allowed. After each turn, the agent assigns one (virtual) point to each of his strategies which would have predicted the correct outcome. At each turn of the game, the agent uses the most successful strategy, i.e. the one with the most virtual



points, among his  $s$  strategies. The strategy-space  $\mathcal{V}_m$  forms a  $2^m$ -dimensional hypercube with strategies at the  $2^{2^m}$  vertices. Fortunately, the MG's standard deviation is essentially unchanged if a 'reduced' strategy space  $\mathcal{U}_m$  is used instead of  $\mathcal{V}_m$  [3]: the  $\mathcal{U}_m$  only contains  $2^{m+1}$  strategies or equivalently  $2^m$  strategy pairs  $\{\mathcal{G}\}$ . The two strategies within a given pair  $\mathcal{G}$  are anticorrelated, i.e. they differ by the maximum possible Hamming distance  $d_H = 2^m$  [3]. Strategies between any two pairs  $\mathcal{G}$  and  $\mathcal{G}'$  are uncorrelated, i.e. they differ by  $d_H = 2^{m-1}$ . The results presented in this paper employ the reduced strategy space.

If  $n_R$  agents use the same strategy  $R$ , then they will act as a 'crowd', i.e. they will make the same decision. If  $n_{\bar{R}}$  agents simultaneously use the anticorrelated strategy  $\bar{R}$ , they will make the opposite decision and will hence act as an 'anticrowd' ( $\mathcal{G} \equiv (R, \bar{R})$ ). If  $n_R \approx n_{\bar{R}}$  for all  $\mathcal{G}$ , then the actions of the crowds and anticrowds cancel and the standard deviation  $\sigma$  of the number of agents choosing a given option (the so-called 'attendance' time-series  $A(t)$ ) will be small. In contrast if  $n_R \gg n_{\bar{R}}$  for all  $\mathcal{G}$ , then  $\sigma$  will be large. Since there is no correlation between  $\mathcal{G}$  and  $\mathcal{G}'$ , each group  $\mathcal{G}$  comprising a crowd-anticrowd pair  $(n_R, n_{\bar{R}})$  will contribute to the attendance  $A(t)$  via a separate random walk in time of step-size  $|n_R - n_{\bar{R}}|$ . The variances of these walks can then be summed to obtain the standard deviation of  $A(t)$ . Hence we describe the MG standard deviation using the following theoretical expression:

$$\sigma = \left[ \sum_{\mathcal{G}=1}^{2^m} \sigma_{\mathcal{G}}^2 \right]^{\frac{1}{2}} = \left[ \sum_{\mathcal{G} \equiv (R, \bar{R})} \frac{1}{4} |n_R - n_{\bar{R}}|^2 \right]^{\frac{1}{2}} \quad (1)$$

where both time-averaging, for a given configuration of initial strategies, and configuration-averaging have been carried out. We now demonstrate that this crowd-anticrowd cancellation underlies the numerical results for  $\sigma$  vs.  $m$  [4]. We run the numerical simulation of the MG and wait until transients in  $A(t)$  have disappeared. At timestep  $t_0$ , we read out the number of players playing each strategy  $R$ , where  $R = 1, 2, \dots, 2^{m+1}$ . For each strategy pair  $\mathcal{G} = (R, \bar{R})$ , we calculate  $n_R - n_{\bar{R}}$  at time  $t_0$  and hence obtain  $\sigma$ . We then average this  $\sigma$  over 1000 timesteps to simulate the time-averaging. We have checked that our results are insensitive to the precise time-averaging procedure. Finally, we average over 16 runs to simulate the configuration-averaging. Figure 1 compares the resulting time and configuration-averaged  $\sigma$



with that obtained from the numerical MG simulation. The agreement is very good for all  $m$ ,  $s$  and  $N$  (not shown). We conclude that the crowd-anticrowd cancellation can indeed quantitatively explain the numerical results of Ref. [4].

We need expressions for the number of agents using each strategy, i.e.  $\{n_R\}$ . Since the labels  $R$  are arbitrary in Eq. (1), the ordering of strategies  $\{n_R\}$  has no particular significance. At any particular time  $t_0$ , these  $2^{m+1}$  strategies can be ranked according to their virtual points by a sort-operation  $\Theta$  acting on the list  $\{n_R\}$ . Hence  $\{n_R\} \xrightarrow{\Theta} \{n_\rho\}$  where  $\rho$  is the virtual-point rank label with  $\rho = 1$  being the highest scoring strategy and  $\{n_\rho\} \equiv n_{\rho=1}, n_{\rho=2}, \dots, n_{\rho=2^{m+1}}$ . Another useful method of ordering is achieved by ranking strategies  $\{n_R\}$  at time  $t_0$  according to their popularity. In particular, strategy  $r = 1$  is defined as the strategy which is being used by the largest number of agents, strategy  $r = 2$  is being used by the second largest number of agents, etc. We denote the popularity ranking by  $\{n_r\}$  where  $\{n_r\} \equiv n_{r=1}, n_{r=2}, \dots, n_{r=2^{m+1}}$ . Note that  $\{n_r\}$  can be obtained from  $\{n_R\}$  and hence  $\{n_\rho\}$  by sort operations, i.e.  $\{n_R\} \xrightarrow{\Psi} \{n_r\}$  and hence  $\{n_\rho\} \xrightarrow{\Gamma} \{n_r\}$ . Each agent plays the available strategy having highest virtual points; this allows an analytic expression to be obtained for the probability that an agent plays a given strategy for general  $s$ , which in turn yields

$$n_r = N \left( \left[ 1 - \frac{(r-1)}{2^{m+1}} \right]^s - \left[ 1 - \frac{r}{2^{m+1}} \right]^s \right) \quad (2)$$

where  $\sum_{r=1}^{2^{m+1}} n_r = N$  as required. Since agents are discrete objects, the simulation tends to produce discrete steps in the curves of  $n_r$  as a function of  $r$ . This effect becomes more pronounced as  $m$  increases since the total number of strategies  $2^{m+1}$  then exceeds the population size  $N$ . We therefore convert the theoretical  $n_r$  values of Eq. (2) to an integer. For large  $m$  such that  $2^{m+1} \gg N$ , the resulting theoretical values are typically  $n_r \sim 1$  for small  $r$  and  $n_r = 0$  for  $r > N$ . Figure 2 compares the theoretical values of  $n_r$  calculated using Eq. (2) with  $s = 2$  and  $N = 101$ , to numerical values taken from the MG simulation. The agreement is good.

For the virtual-point ordered list  $\{n_\rho\}$ , the strategy  $\rho' = 2^{m+1} + 1 - \rho$  is *always* anticor-



related to the strategy  $\rho$ , i.e.  $\rho' \equiv \bar{\rho}$ . Hence knowledge of the sort operation  $\Gamma$  completely determines where each strategy's anticorrelated partner is located in the popularity-ordered list  $\{n_r\}$ . Since we are here only interested in time-averaged and run-averaged  $\sigma$ , we only need to consider the probability distribution of locations of  $\bar{r}$  in the popularity-ordered list  $\{n_r\}$ . We therefore replace the sort operation  $\Gamma$  by a probability function  $P(r' = \bar{r})$  which gives the probability that any strategy  $r'$  is the anti-correlated partner of strategy  $r$  in the list  $\{n_r\}$ . Hence Eq. (1) becomes

$$\sigma = \left[ \frac{1}{2} \sum_{r=1}^{2^{m+1}} \sum_{r'=1}^{2^{m+1}} \frac{1}{4} |n_r - n_{r'}|^2 P(r' = \bar{r}) \right]^{\frac{1}{2}} \quad (3)$$

where the factor  $\frac{1}{2}$  discounts double-counting. There are two limiting cases. When the virtual-point ordered list  $\{n_\rho\}$  and the popularity-ordered list  $\{n_r\}$  are identical, then  $P(r' = \bar{r})$  will be a  $\delta$ -function at  $r' = 2^{m+1} + 1 - r$  and hence Eq. (3) has the same form as Eq. (1). In the opposite case where the two ordered lists are uncorrelated,  $P(r' = \bar{r})$  should be a flat distribution. In each case, Eqs. (2) and (3) can be combined to obtain closed-form analytic solutions for arbitrary  $s$  and  $N$ .

Figure 3 shows  $P(r' = \bar{r})$  for  $r = 1$  as a function of  $r'$ , taken from the numerical MG simulation at  $m = 2, 5$  and  $10$ . For small  $m$  ( $m = 2$ ) the anticorrelated strategy to the most popular strategy (i.e.  $r = 1$ ) is at  $r' = 2^{m+1}$ , i.e. it is the least popular strategy. Hence  $P(r' = \bar{r})$  resembles the  $\delta$ -function limiting case mentioned above. From Fig. 2 we know that very few agents will therefore pick this anticorrelated strategy. Hence the crowd-anticrowd cancellation will be small and  $\sigma$  will be large, as can be seen in Fig. 1. As  $m$  increases ( $m = 5$ ) a remarkable effect occurs: the peak in  $P(r' = \bar{r})$  moves up toward  $r = 1$ . Hence both  $r = 1$  and its anticorrelated partner  $\bar{r}$  are now very popular. Whereas for  $m = 2$  it seemed like there was an effective 'repulsion' between  $r$  and  $\bar{r}$ , for  $m = 5$  this now seems more like an attraction. Amusingly, the shape of  $P(r' = \bar{r})$  for  $m = 5$  is reminiscent of the screening effect of a negative charge cloud around a positive charge placed at  $r = 1$ , or even a bound electron-hole pair (i.e. exciton) with the crowd (anticrowd) playing the role of the positive (negative) charge. For large  $m$  ( $m = 10$ ), the ability of the anticrowd to 'screen'



the crowd has decreased yielding a rather flat distribution as shown. The consequence of this strong crowd-anticrowd correlation which appears as  $m$  increases, is that the crowd and anticrowd become comparable in size. Hence  $\sigma$  is small for  $m \sim 5 - 6$ , in agreement with Fig. 1. Note that the MG cannot fully ‘optimize’ itself by building equal-sized crowds and anticrowds. In modified models of MG [11] where this in-built frustration in the strategy space is allowed to relax, equal-sized crowds and anti-crowds do naturally emerge.

We now consider analytic expressions for  $\sigma$  for general  $s$  using Eqs. (2) and (3). For small  $m$ , the virtual-point ordered list  $\{n_\rho\}$  and the popularity-ordered list  $\{n_r\}$  will be very similar, hence  $P(r' = \bar{r}) \sim \delta_{r', 2^{m+1}+1-r}$ . The discreteness of the agents will be unimportant since  $n_r \gg 1$ , hence  $n_r$  can be treated as continuous. Equations (2) and (3) yield

$$\sigma_{\text{low } m} = \frac{N}{2} \left[ \sum_{r=1}^{2^m} \left[ \left(1 - \frac{r-1}{2^{m+1}}\right)^s - \left(1 - \frac{r}{2^{m+1}}\right)^s - \left(\frac{r}{2^{m+1}}\right)^s + \left(\frac{r}{2^{m+1}}\right)^s \left(1 - \frac{1}{r}\right)^s \right]^2 \right]^{\frac{1}{2}}. \quad (4)$$

For  $s = 2$  this becomes

$$\sigma_{\text{low } m} = \frac{N}{\sqrt{3} 2^{\frac{m}{2}+1}} \left[ 1 - 2^{-2(m+1)} \right]^{\frac{1}{2}}. \quad (5)$$

Figure 4 shows these analytic curves for  $s = 2$  and  $s = 4$  (solid lines monotonically decreasing). As might be expected using the extreme  $\delta$ -function form for  $P(r' = \bar{r})$ , these curves are slightly higher than the numerical results in Fig. 1 for small  $m$ . Now consider the opposite extreme of uncorrelated  $r'$  and  $\bar{r}$ , i.e. the flat distribution  $P(r' = \bar{r}) \sim 2^{-(m+1)}$ . For  $s = 2$  this gives

$$\sigma_{\text{low } m} = \frac{N}{\sqrt{3} 2^{\frac{(m+3)}{2}}} \left[ 1 - 2^{-2(m+1)} \right]^{\frac{1}{2}}. \quad (6)$$

Equation (6) typically produces lower estimates for each  $s$  value (dashed lines). Values of  $\sigma$  obtained from separate numerical runs tend to be scattered in the region of these curves. For larger  $m$  ( $m > 6$ ) we cannot ignore the discreteness of the agents (Fig. 2). In this regime,  $n_r \sim 1$  for  $r < N$  while  $n_r = 0$  for  $r > N$ . Using the integer form of Eq. (2) for high  $m$ , and the flat distribution for  $P(r' = \bar{r})$ , yields



$$\sigma_{\text{high } m} = \frac{\sqrt{N}}{2} \left[ 1 - \frac{N}{2^{m+1}} \right]^{\frac{1}{2}}. \quad (7)$$

This expression approaches the coin-toss limit from below as  $m \rightarrow \infty$ , as shown in Fig. 4 (solid line monotonically increasing). Within the approximation used here, this curve is insensitive to  $s$ . Note that the numerical MG results are also consistent with this finding of weak  $s$ -dependence for large  $m$ . Hence Fig. 4 gives a clear picture of what happens to  $\sigma$  as  $s$  and  $m$  increase: considering the monotonically decreasing curve for low  $m$ , and the monotonically increasing curve for high  $m$ , we see that (a) there should be a minimum in  $\sigma$  for  $s = 2$ , (b) this minimum should move to higher  $m$  as  $s$  increases and (c) the minimum should become shallower as  $s$  increases. Each of these statements agrees with numerical simulation results (c.f. Fig. 1). In addition, the curve  $\sigma_{\text{high } m} \rightarrow 0$  at  $N = 2^{m+1}$ , i.e. for  $m \sim 5 - 6$ .

We now compare the theoretical crowd-anticrowd calculation (Eq. (3)) with the numerical simulation. The most interesting case is  $s = 2$ . Figure 5 shows the spread of numerical values for different runs (open circles) compared to theory (solid circles). The agreement is good. The appropriate analytic expressions for the probability function  $P(r' = \bar{r})$  in Eq. (3) involve multiple sums and are complicated: we therefore obtained the results for each  $m$  in Fig. 5 by generating the corresponding  $P(r' = \bar{r})$  forms to those in Fig. 3. The theoretical points tend to lie in between the limiting curves of Fig. 4 *except* at the minimum where the remarkable form for  $P(r' = \bar{r})$  (recall Fig. 3) pulls the theoretical value slightly below the cruder limiting curves of Fig. 4. Elsewhere we will discuss simpler closed-form expressions for  $P(r' = \bar{r})$ , together with an analysis of the time-series  $A(t)$  and history bit-string statistics within the crowd-anticrowd model [12].

In summary we presented an analytic analysis of crowding effects in MG which offers a novel explanation of the main finding of Ref. [4]. Future work will explore possible connections to the spin-glass formalism of Challet *et al* [5].

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## REFERENCES

- [1] W.B. Arthur, Science **284**, 107 (1999).
- [2] H.E. Stanley, Computing in Science & Engineering Jan/Feb, 76 (1999).
- [3] D. Challet and Y.C. Zhang, Physica A **246**, 407 (1997); *ibid.* **256**, 514 (1998); *ibid.* **269**, 30 (1999).
- [4] R. Savit, R. Manuca and R. Riolo, Phys. Rev. Lett. **82**, 2203 (1999).
- [5] D. Challet and M. Marsili, Phys. Rev. E **60**, R6271 (1999); D. Challet, M. Marsili, and R. Zecchina, Phys. Rev. Lett. **84**, 1824 (2000); D. Challet and M Marsili, cond-mat/9908480.
- [6] A. Cavagna, I. Giardina, J.P. Garrahan and D. Sherrington, Phys. Rev. Lett. **83**, 4429 (1999).
- [7] R. D'Hulst and G.J. Rodgers, Physica A **270**, 514 (1999).
- [8] A. Cavagna, Phys. Rev. E **59**, R3783 (1999).
- [9] N.F. Johnson, P.M. Hui, D. Zheng and M. Hart, J. Phys. A: Math. Gen. **32** L427 (1999).
- [10] N.F. Johnson, M. Hart and P.M. Hui, Physica A **269**, 1 (1999). This conference paper provides a preliminary report of the qualitative effects of crowding.
- [11] N.F. Johnson, P.M. Hui, R. Jonson and T.S. Lo, Phys. Rev. Lett. **82**, 3360 (1999).
- [12] P. Jefferies, M. Hart, N.F. Johnson and P.M. Hui, in preparation.



## FIGURES

FIG. 1. Standard deviation  $\sigma$  for the Minority Game as a function of memory-size  $m$  for  $s = 2, 3, 4$  strategies per agent and  $N = 101$  agents. Solid curve: numerical simulation. Dashed curve: crowd-anticrowd theory using Eq. (1). Random (coin-toss) limit  $\sigma = \sqrt{N}/2 = 5.0$  is indicated.

FIG. 2. Histograms of the number of agents using strategy  $r$  for  $r = 1$  (most popular) to  $r = 2^{m+1}$  (least popular). Results are shown for  $m = 2$  (left scale) and  $m = 7$  (right scale). Solid lines: numerical simulation. Dashed lines: theory from Eq. (2).

FIG. 3. Probability function  $P(r' = \bar{r})$  giving the probability that the strategy ranked  $r'$  on the popularity-ordered list, is anti-correlated with the strategy ranked  $r$ . Results are shown for  $r = 1$  (i.e. most popular strategy) as a function of  $r'$  for  $m = 2$  (dotted-dashed),  $m = 5$  (dotted) and  $m = 10$  (solid).  $s = 2$  and  $N = 101$ . Note that  $\sum_{r'} P(r' = \bar{r}) = 1$ .

FIG. 4. Theoretical curves for  $\sigma$  using Eqs. (4)-(7). Monotonically decreasing curves for  $s = 2, 4$  at low  $m$  ( $m < 6$ ): solid lines correspond to  $\delta$ -function  $P(r' = \bar{r})$  distribution neglecting agent discreteness, dashed lines correspond to flat distribution neglecting agent discreteness. Monotonically increasing solid line for large  $m$  ( $m > 6$ ) is independent of  $s$ : it corresponds to a flat distribution and accounts for agent discreteness.

FIG. 5. Theoretical crowd-anticrowd calculation (solid circles) vs. numerical simulations (open circles) for  $s = 2$ ,  $N = 101$ . 16 numerical runs are shown for each  $m$ .



















